Lecture 13

Review

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$.

- The angle between $\vec{u}$ & $\vec{v}$ is

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) \iff \cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$

- The projection of $\vec{u}$ onto $\vec{v}$ is

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$$

- The area of the parallelogram spanned by $\vec{u}$ & $\vec{v}$ is

$$A = |\vec{u}\times\vec{v}|$$

- The area of the triangle with vertices P, Q, R is

$$A = \frac{1}{2}|\vec{PQ}\times\vec{PR}|$$

- The volume of the parallelpiped spanned by $\vec{u}, \vec{v}, \& \vec{w}$ is

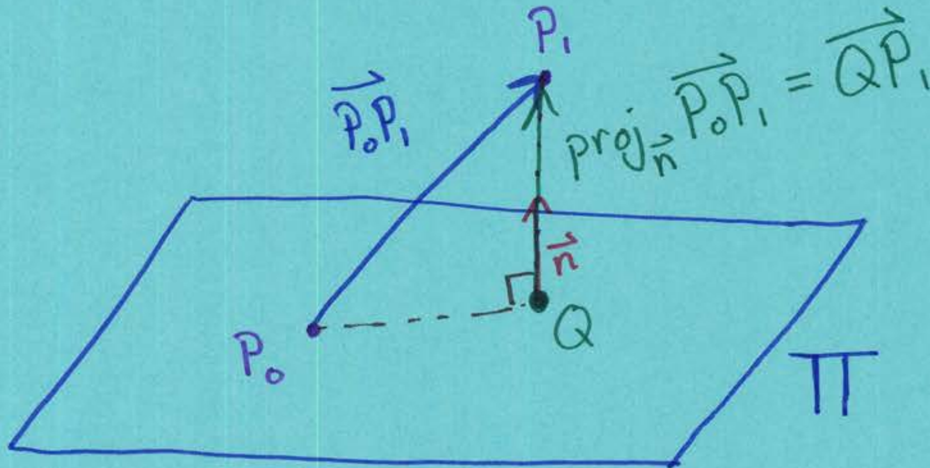
$$V = |\vec{u}\cdot(\vec{v}\times\vec{w})|$$

- Given a vector $\vec{v} = \langle a, b, c \rangle$ and a point $P = (x_0, y_0, z_0)$ the equation for the line through P with direction $\vec{v}$ is:
  - vector eqn: $\vec{r}(t) = \vec{v}\,t + \overrightarrow{OP} = \langle at + x_0, bt + y_0, ct + z_0 \rangle$
  - parametric eqns: $x = at + x_0, \quad y = bt + y_0, \quad z = ct + z_0$
  - symmetric eqns: $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$

- Given a vector $\vec{n} = \langle a, b, c \rangle$ and a point $P_0 = (x_0, y_0, z_0)$, the equation for the plane through P_0 with normal vector $\vec{n}$ is:
  - vector eqn: $\vec{n} \cdot (\overrightarrow{P_0 P}) = \vec{n} \cdot \big((x, y, z) - (x_0, y_0, z_0) \big)$
    $= \vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$
  - scalar eqn: $a(x - x_0) + b(y - y_0) + c(z - z_0) = ax + by + cz + d = 0$

The distance between a plane passing through P_0 with normal vector $\vec{n}$ to a point $P_1 = (x_1, y_1, z_1)$ is:

$$D = \left| \mathrm{proj}_{\vec{n}} \overrightarrow{P_0 P_1} \right| = \frac{|ax_1 + by_1 + cz_1 + d|}{|\vec{n}|}$$



D is the distance from Q, the point on Π closest to P_1, to P_1, i.e., $D = |\overrightarrow{QP_1}|$.

- The tangent line to a curve $\vec{r}(t)$ at a point $\vec{r}(t_0)$ on the curve is:

$$\vec{l}(t) = \vec{r}\,'(t_0)\,t + \vec{r}(t_0).$$

- Unit tangent vector : $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

- Unit normal & binormal
  - $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$ & $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$
  - $\vec{N}(t) = \vec{B}(t) \times \vec{T}(t)$ & $\vec{B}(t) = \dfrac{\vec{r}\,'(t) \times \vec{r}\,''(t)}{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}$

- Normal plane has vector $/\!/$ $\vec{T}(t)$ (e.g., $\vec{r}\,'(t)$) as perpendicular vector.

- Osculating plane has vector // $\vec{B}(t)$ (e.g., $\vec{r}'(t)\times\vec{r}''(t)$) as perpendicular vector

- Curvature of $\vec{r}(t)$

$$\kappa(t)=\frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

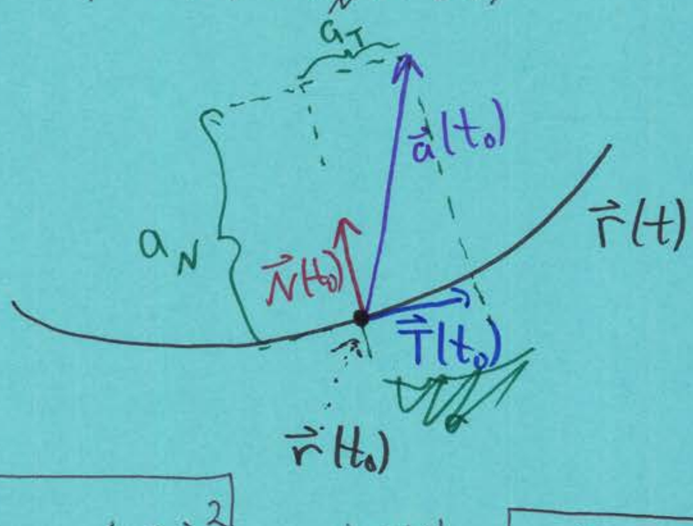
- Curvature of a plane curve $y=f(x)$.
  First parametrize as $\vec{r}(x)=x\hat{i}+f(x)\hat{j}$, then use above formula.

- Tangential (a_T) and normal (a_N) components of acceleration.

$$\vec{a}(t)=a_T\vec{T}(t)+a_N\vec{N}(t)$$

$v(t)=|\vec{r}'(t)|$: speed

$\kappa(t)$= curvature



$$\boxed{a_T=v',\ a_N=\kappa v^2},\quad |\vec{a}(t_0)|=\sqrt{a_T^2+a_N^2}$$

$$a_T=\frac{\vec{r}'(t_0)\cdot\vec{r}''(t_0)}{|\vec{r}'(t_0)|},\quad a_N=\frac{|\vec{r}'(t_0)\times\vec{r}''(t_0)|}{|\vec{r}'(t_0)|}$$